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From Boltzmann transport equation to single-phase-lagging heat conduction

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ABSTRACT

In the present work the single-phase-lagging heat conduction model is re-derived analytically from the Boltzmann transport equation. In contrast to the Maxwell–Cattaneo law (CV model), it is Galilean invariant in the moving media. Based on this model, the governing equation of the microscale heat conduction is established, which is formulated into a delay partial differential equation. The corresponding initial and boundary conditions are prescribed. The thermal oscillation of the single-phase-lagging heat conduction is investigated.

1. Introduction

Consider the classical Fourier law in a homogeneous and isotropic thermally conducting medium

$$\mathbf{q}(\mathbf{r},t) = -k\nabla T(\mathbf{r},t), \tag{1}$$

where the temperature gradient $\nabla T(\mathbf{r},t)$ is a vector function of the position vector $\mathbf{r}$ and the time variable t, the vector $\mathbf{q}(\mathbf{r},t)$ is called the heat flux, k is the thermal conductivity of the material. This classical law has been widely and successfully applied to the conventional engineering heat conduction problems, in which the system has large spatial dimension and the emphasis is on the long time behavior. However, it leads to the infinite speed of heat propagation, implying that a thermal disturbance applied at a certain location in a heat conduction medium can be sensed immediately anywhere else in the medium. This is unaccepted in the transient behavior at extremely short time, say, on the order of picoseconds to femtoseconds. An example is the ultrafast laser heating in thermal processing of materials.

Experimentally it is also shown that the propagation of second sound, ballistic phonon propagation and phonon hydrodynamics in solids at low temperatures depart significantly from the usual parabolic description [1]. With the advances of modern microfabrication technology, more and more microdevices with micro- and nano-scale dimension emerge in various micromechanical systems. The understanding of the microscale heat transport phenomena is critical for the further development of the nanotechnology, especially for the cooling of the large scale integrate circuit. However, the traditional Fourier law leads to the unaccepted result for the microscale heat conduction [2–4]. Many phenomena in the discrete systems including the low-dimensional lattices also challenge the validity of the classical Fourier law [5–7].

Much effort has already been devoted to the modification of the classical Fourier law, which leads to many non-Fourier laws. The most famous one among them is the CV model proposed by Cattaneo and Vernotte [8–10]:

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T, \tag{2}$$

where τ is the time delay. The CV model gives rise to a wave type of heat conduction equation called the hyperbolic heat conduction equation [11]. The natural extension of this model is

$$\mathbf{q}(\mathbf{r},t+\tau) = -k\nabla T(\mathbf{r},t), \tag{3}$$

which was proposed by Tzou [12–16]. The constitutive relation (3) is called the single-phase-lagging (SPL) heat conduction model. The model (3) was further extended to the dual-phase-lagging (DPL) model by Tzou and formulated mathematically as follows [1,17–19]:

$$\mathbf{q}(\mathbf{r},t+\tau_q) = -k\nabla T(\mathbf{r},t+\tau_T), \tag{4}$$

where τ_T and τ_q are the phase lags of the temperature gradient and the heat flux vector, respectively. The first order Taylor expansion of Eq. (4) gives

$$q(\mathbf{r},t) + \tau_q\frac{\partial \mathbf{q}}{\partial t}(\mathbf{r},t) \cong -k\left\{\nabla T(\mathbf{r},t) + \tau_T\frac{\partial}{\partial t}[\nabla T(\mathbf{r},t)]\right\}, \tag{5}$$

which leads to the following governing equation of the temperature field:

$$T_t(\mathbf{r},t) + \tau_q T_{tt}(\mathbf{r},t) = \alpha(\Delta T(\mathbf{r},t) + \tau_T\Delta T_t(\mathbf{r},t)), \tag{6}$$

where the subscript t indicates the partial derivative with respect to time.

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Nomenclature

f	distribution function	*Greek symbols*	
k	thermal conductivity	τ	relaxation time
$\mathbf{r}$	position vector	τ_q	phase lag of the heat flux vector
$\mathbf{q}$	heat flux	τ_T	phase lag of the temperature gradient
t	time variable		
T	temperature field		
$\mathbf{v}$	velocity vector		

Eq. (6) plays a significant role in the investigation of the microscale heat conduction. Firstly it is a unified form of the energy equations of the phonon–electron interaction model [20] and the phonon scattering model [11,21]. These two models have been developed in examining energy transport involving the high-rate heating in which the non-equilibrium thermodynamic transition and the microstructural effect become important associated with shortening of the response time [1,22]. The high-rate heating is developing rapidly due to the advancement of high-power short-pulse laser technologies [23–27]. In addition to its application in the ultrafast pulse-laser heating, the microscale heat conduction equation (6) also arises in describing and predicting phenomena such as temperature pulses propagating in superfluid liquid helium, nonhomogeneous lagging response in porous media, thermal lagging in amorphous materials, and effects of material defects and thermo-mechanical coupling, etc. [1]. The study of Eq. (6) is thus of considerable importance in understanding and applying these rapidly emerging technologies. We have examined its well-posedness [28,29] and investigated the thermal oscillation and resonance phenomena in detail [30] which are believed to be a manifestation of non-equilibrium behavior of microscale heat conduction [22].

Unfortunately, it was shown that the CV model violates the Galilean principle of relativity [31], thus it cannot be applied to the moving medium. Therefore, it is desirable to examine whether the model (3) suffers from the same drawback.

The Boltzmann transport equation (BTE) is a fundamental equation in statistical physics for describing the non-equilibrium phenomena. Therefore, many efforts are dedicated to establishing the non-Fourier laws from the BTE. The phonon–electron interaction model [20] was developed from BTE on a quantum mechanical and statistical basis. A phonon radiative transport equation between two parallel plates was established from the BTE for the heat transport in dielectric solid films [3]. Based on the BTE, Chen [32,33] proposed a ballistic–diffusive heat conduction model of microscale heat transport in devices where the characteristic length is comparable to the mean free path of the energy-carrier and/or the characteristic time is comparable to the relaxation time of the energy-carrier. The classical Fourier law and CV model were also re-established from the BTE [4]. Recently, we re-derived the dual-phase-lagging heat conduction model (4) from the discrete form of the BTE [34]. In the present work the methodology in [34] is developed to re-establish the SPL heat conduction model (3) from the BTE in the partial differential equation form.

Finally, the governing equation of the SPL heat conduction, which is expressed as the delay partial differential equation, is obtained by combining Eq. (3) with the energy balance equation. The associated initial and boundary conditions for this equation are prescribed. The thermal oscillation phenomenon is investigated.

2. Examination of SPL model by Galilean principle of relativity

In [31], it was found that the CV model is not Galilean invariant. In this section we attempt to examine the SPL model (3). Consider the following Galilean transformation and some notations:

$$\mathbf{r}' = \mathbf{r} - \mathbf{U}t, \quad t = t, \quad \theta(\mathbf{r}', t) = T(\mathbf{r}', t), \quad \mathbf{q}'(\mathbf{r}', t) = \mathbf{q}(\mathbf{r}', t), \tag{7}$$

where $\mathbf{U}$ is the constant velocity between two inertial reference frames. From the first relation in Eq. (7), it is evident that $\nabla_{\mathbf{r}'} = \nabla_{\mathbf{r}}$. Therefore, we have

$$\nabla_{\mathbf{r}'}\theta(\mathbf{r}', t) = \nabla_{\mathbf{r}} T(\mathbf{r}, t). \tag{8}$$

Subsequently, Eq. (3) becomes

$$\mathbf{q}'(\mathbf{r}', t + \tau) = -k\nabla_{\mathbf{r}'}\theta(\mathbf{r}', t). \tag{9}$$

The observation shows that Eq. (9) has the same form as Eq. (3) and it does not involve the velocity $\mathbf{U}$. Thus the SPL heat conduction model is invariant under the Galilean transformation (7) and can be employed to study the microscale heat conduction problems in moving media. Therefore, compared with the CV model, it has the obvious advantage.

Note that the first order Taylor expansion of the left side of Eq. (3) with respect to the time variable gives rise to the CV model (2) which violates the Galilean principle of relativity. Then one natural question is whether the higher order approximation of the left side of Eq. (3) would lead to the Galilean invariant heat conduction models. In order to address this question, we first consider the following heat conduction model with the lagging behavior:

$$\mathbf{q}(\mathbf{r}, t) + \tau\frac{\partial \mathbf{q}(\mathbf{r}, t)}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2 \mathbf{q}(\mathbf{r}, t)}{\partial t^2} = -k\nabla T, \tag{10}$$

which is obtained by the second order approximation of the left side of Eq. (3). By the Galilean transformation (7), we have

$$\frac{\partial \mathbf{q}(\mathbf{r}, t)}{\partial t} = \frac{\partial \mathbf{q}'(\mathbf{r}', t)}{\partial t} - \frac{\partial \mathbf{q}'(\mathbf{r}', t)}{\partial \mathbf{r}'}\cdot\mathbf{U}, \tag{11}$$

$$\frac{\partial^2 \mathbf{q}(\mathbf{r}, t)}{\partial t^2} = \frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial t^2} - 2\frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial t\partial \mathbf{r}'}\cdot\mathbf{U} + \mathbf{U}\cdot\frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial^2 \mathbf{r}'}\cdot\mathbf{U}. \tag{12}$$

Substituting Eqs. (8), (11) and (12) into Eq. (10) yields

$$\mathbf{q}'(\mathbf{r}', t) + \tau\left[\frac{\partial \mathbf{q}'(\mathbf{r}', t)}{\partial t} - \frac{\partial \mathbf{q}'(\mathbf{r}', t)}{\partial \mathbf{r}'}\cdot\mathbf{U}\right] + \frac{\tau^2}{2}\left[\frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial t^2} - 2\frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial t\partial \mathbf{r}'}\cdot\mathbf{U} + \mathbf{U}\cdot\frac{\partial^2 \mathbf{q}'(\mathbf{r}', t)}{\partial^2 \mathbf{r}'}\cdot\mathbf{U}\right] = -k\nabla_{\mathbf{r}'}\theta(\mathbf{r}', t). \tag{13}$$

Note that Eq. (13) depends on the constant velocity $\mathbf{U}$. This indicates that the constitutive relation (10) is not independent on the observer's speed, therefore, violates the Galilean principle of relativity. Similar deductions show that the other higher order approximations of the SPL heat conduction model (3) suffer from the same drawback. From the above derivation, one can see that it is the presence of the time partial derivative in the constitutive relation that leads to the violation of the Galilean principle of relativity.

3. Boltzmann transport equation and SPL model

3.1. Boltzmann transport equation

In the absence of external forces, the Boltzmann transport equation reads

$$\frac{\partial f(\mathbf{r},\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\nabla f(\mathbf{r},\mathbf{v},t)=\left(\frac{\delta f}{\delta t}\right)_{\text{coll}}, \tag{14}$$

where $f(\mathbf{r},\mathbf{v},t)$ is the probability density of finding a classical pointlike particle at position $\mathbf{r}$ and time t with speed $\mathbf{v}$, $(\delta f/\delta t)_{\text{coll}}$ represents the rate of change of $f(\mathbf{r},\mathbf{v},t)$ due to collisions. In the present work the pointlike particles refer to the heat carriers such as the electrons and phonons. Eq. (14) is obviously incomplete, since the precise form of collision term is not known. A very simple method for taking into account collision effects is the relaxation model. In this model it is assumed that the effect of collisions is to restore a situation of local equilibrium, characterized by a distribution function $f_0(\mathbf{r},\mathbf{v})$. It suggests that a situation initially not in equilibrium, described by a distribution function $f(\mathbf{r},\mathbf{v},t)$ different from $f_0(\mathbf{r},\mathbf{v})$, reaches a local equilibrium condition exponentially, as a result of collisions, with a relaxation time τ. This relaxation time is of the order of the time between collisions and may also be written as ν^{-1} where ν represents the relaxation collision frequency. This model can be expressed mathematically as

$$\left(\frac{\delta f}{\delta t}\right)_{\text{coll}}=-\frac{(f-f_0)}{\tau}. \tag{15}$$

Therefore, Eq. (14) becomes

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f=-\frac{(f-f_0)}{\tau}. \tag{16}$$

3.2. Derivation of SPL model

By using the definition of $f(\mathbf{r},\mathbf{v},t)$, the energy flux vector $\mathbf{q}$ can be expressed as

$$\mathbf{q}(\mathbf{r},t)=\int_{\epsilon}\mathbf{v}(\mathbf{r},t)f(\mathbf{r},\epsilon,t)\epsilon D(\epsilon)\mathrm{d}\epsilon, \tag{17}$$

where ϵ is the kinetic energy, $D(\epsilon)$ is the density of states. The details of the derivation of Eq. (17) can be found in [34].

By using difference to approximate the time derivative in Eq. (16), we have

$$\frac{f(\mathbf{r},\epsilon(\mathbf{v}),t+\triangle t)-f(\mathbf{r},\epsilon(\mathbf{v}),t)}{\triangle t}+\mathbf{v}\cdot\nabla f(\mathbf{r},\mathbf{v},t)=-\frac{f(\mathbf{r},\epsilon(\mathbf{v}),t)-f_0(\mathbf{r},\mathbf{v})}{\tau}. \tag{18}$$

Setting $\triangle t=\tau$, yields

$$\frac{f(\mathbf{r},\epsilon(\mathbf{v}),t+\tau)-f(\mathbf{r},\epsilon(\mathbf{v}),t)}{\tau}+\mathbf{v}\cdot\nabla f(\mathbf{r},\epsilon(\mathbf{v}),t)=\frac{f_0(\mathbf{r},\mathbf{v})-f(\mathbf{r},\varepsilon(\mathbf{v}),t)}{\tau}. \tag{19}$$

Rearranging the terms of the above equation yields

$$\tau\mathbf{v}\cdot\nabla f(\mathbf{r},\epsilon(\mathbf{v}),t)+f(\mathbf{r},\epsilon(\mathbf{v}),t+\tau)=f_0. \tag{20}$$

Multiplying $\varepsilon D(\varepsilon)\mathbf{v}$ on both sides of this equation and integrating over all possible energies give

$$\int_{\varepsilon}\tau\mathbf{v}\cdot\nabla f(\mathbf{r},\varepsilon(\mathbf{v}),t)\mathbf{v}\varepsilon D(\varepsilon)\mathrm{d}\varepsilon+\mathbf{q}(\mathbf{r},\mathbf{v},t+\tau)=\mathbf{0}. \tag{21}$$

In deriving this equation, the relation

$$\int_{\varepsilon}f_0\varepsilon D(\varepsilon)\mathbf{v}\,\mathrm{d}\varepsilon=\mathbf{0} \tag{22}$$

has been used. The verification of this assertion was given in [34].

Assume that the relaxation time τ does not depend on the energy of the system and the system has achieved the quasi-equilibrium state. Then $\nabla f=(\mathrm{d}f_0/\mathrm{d}T)\,\nabla T$, thus Eq. (21) becomes

$$\mathbf{q}(\mathbf{r},t+\tau)=-\mathbf{k}\cdot\nabla T(\mathbf{r},t), \tag{23}$$

where $\mathbf{k}$ is the thermal conductivity tensor

$$\mathbf{k}=\int\tau\mathbf{v}\mathbf{v}\frac{\mathrm{d}f_0}{\mathrm{d}T}\varepsilon D(\varepsilon)\mathrm{d}D(\varepsilon). \tag{24}$$

For the isotropic materials, $\mathbf{k}$ takes the form of

$$\mathbf{k}=k\mathbf{I},$$

here $\mathbf{I}$ is the unit matrix and k is a constant. Thus Eq. (23) reduces to the SPL heat conduction model (3). Therefore, the relation (3) is derivable from the Boltzmann transport equation. And the thermal conductivity can be obtained by formula (24).

Note that in order to derive the SPL heat conduction model, Eq. (19) was employed to approximate the Boltzmann transport equation (16) and the relation $\nabla f\approx\nabla f_0$ was assumed in order to obtain Eq. (23) from Eq. (21). These suggest that the non-equilibrium state under consideration is not far from equilibrium.

4. Governing equation of SPL heat conduction

In this section, the constitutive relation (3) is directly employed to establish the governing equation of the SPL heat conduction. A delay SPL heat conduction equation is thus obtained. After the initial and boundary conditions are prescribed, the initial-boundary value problem of the delay SPL heat conduction is formulated.

We start with the following energy balance equation:

$$-\nabla.\mathbf{q}(\mathbf{r},t)+Q(\mathbf{r},t)=C_p\frac{\partial T(\mathbf{r},t)}{\partial t} \tag{25}$$

with C_p being the volumetric heat capacity, Q the volumetric heat source.

For time instant $t+\tau$, Eq. (25) becomes

$$-\nabla\cdot\mathbf{q}(\mathbf{r},t+\tau)+Q(\mathbf{r},t+\tau)=C_p\frac{\partial T(\mathbf{r},t+\tau)}{\partial t}. \tag{26}$$

The divergence of Eq. (3) leads to

$$\nabla\cdot\mathbf{q}(\mathbf{r},t+\tau)=-k\Delta T(\mathbf{r},t), \tag{27}$$

where Δ is Laplacian operator.

Combining Eqs. (26) and (27) gives rise to

$$\Delta T(\mathbf{r},t)+\frac{1}{k}Q(\mathbf{r},t+\tau)=\frac{1}{\alpha}\frac{\partial T(\mathbf{r},t+\tau)}{\partial t}, \tag{28}$$

where $\alpha=\frac{k}{C_p}$. Eq. (28) is the governing equation of the SPL heat conduction based on the constitutive equation (3).

Setting $t'=t+\tau$ in Eq. (28), we have

$$\Delta T(\mathbf{r},t'-\tau)+\frac{1}{k}Q(\mathbf{r},t')=\frac{1}{\alpha}\frac{\partial T(\mathbf{r},t')}{\partial t'}\quad\text{for } t'>\tau. \tag{29}$$

Note that this equation is a delay partial differential equation. Hereafter it is termed as the delay SPL heat conduction equation.

Let R be the heat conduction region and S_i $(i=1,2,3,\ldots,m)$ the boundary surfaces. The boundary condition of Eq. (29) is generally written as

$$k_i\frac{\partial T(\mathbf{r},t')}{\partial n_i}+h_iT(\mathbf{r},t')=f_i(\mathbf{r},t')\quad\text{on the boundary surface } S_i, \tag{30}$$

where constants k_i and h_i satisfy $k_i^2+h_i^2\neq 0$; $\mathbf{n}=(n_1,n_2,n_3)^T$ is the outward normal of surface S. The general boundary condition (30) is called the mixed type boundary condition which commonly occurs when a fluid flows over a solid surface to have a convection boundary condition. It reduces to the specified temperature type boundary condition when k_i $(i=1,2,3,\ldots,m)$ vanish and to the specified heat flux type boundary condition when $h_i=0$ $(i=1,2,3,\ldots,m)$.

The initial condition for Eq. (29) must be prescribed as follows [35]:

$$T(\mathbf{r},t')=\phi(\mathbf{r},t')\quad\text{in the region } R,\ \ t'\in[0,\tau], \tag{31}$$

note that in order to guarantee the uniqueness of the temperature field, the initial temperature field must be given in a time period. This is different from the traditional initial condition for the classical Fourier law. The detail discuss will be given in another paper. If the traditional initial condition is still employed for the delay partial differential equation (29), that is, the distribution of temperature field is specified only at $t = \tau$, then we must set the temperature vanish in the time interval $[0, \tau)$.

By using the relation between t and t', we have

$$T(\mathbf{r}, t) = \phi(\mathbf{r}, t) \quad \text{in the region } R \text{ for } t \in [-\tau, 0]. \tag{32}$$

When the lagging time τ is sufficiently small, initial condition (32) is equivalent to specify all the time derivatives of temperature field at the initial moment, namely,

$$\begin{aligned} &T(\mathbf{r}, t)|_{t=0} = T_0(\mathbf{r}), \quad \left.\frac{\partial T(\mathbf{r}, t)}{\partial t}\right|_{t=0} = T_1(\mathbf{r}), \\ &\left.\frac{\partial^2 T(\mathbf{r}, t)}{\partial t^2}\right|_{t=0} = T_2(\mathbf{r}), \ldots, \left.\frac{\partial^n T(\mathbf{r}, t)}{\partial t^n}\right|_{t=0} = T_n(\mathbf{r}), \ldots, \end{aligned} \tag{33}$$

where $T_n(\mathbf{r})$ $(n = 0, 1, 2, \ldots)$ are given functions of the position vector $\mathbf{r}$.

5. Thermal oscillation of SPL heat conduction

Thermal oscillation is a distinct feature of microscale heat conduction. In [36], the damping and resonance characteristics of thermal waves were investigated under the CV model. In [30], based on the approximate dual-phase-lagging model (5), the thermal oscillation phenomena were explored in details. Interestingly, Tzou [1] and Vadasz [37–40] developed an approximate equivalence between the heat conduction in porous media and the dual-phase-lagging heat conduction. Thus the possible occurrence of thermal oscillation in porous media was examined [37–40]. In the present work, based on the model (3) the thermal oscillation phenomenon of the microscale heat conduction is investigated.

5.1. Occurring condition for thermal oscillation

Consider the following single-phase-lagging heat conduction problem without heat source:

$$\Delta T(\mathbf{r}, t' - \tau) = \frac{1}{\alpha}\frac{\partial T(\mathbf{r}, t')}{\partial t'}, \quad t' > \tau, \tag{34}$$

$$k\frac{\partial T(\mathbf{r}, t')}{\partial n} + hT(\mathbf{r}, t') = 0, \tag{35}$$

$$T(\mathbf{r}, t') = \phi(\mathbf{r}, t'), \quad t' \in [0, \tau]. \tag{36}$$

Setting

$$T(\mathbf{r}, t') = X(\mathbf{r})\Gamma(t') \tag{37}$$

and substituting Eq. (37) into Eq. (34) yield

$$\frac{1}{\alpha}\frac{\Gamma_{t'}(t')}{\Gamma(t' - \tau)} = \frac{\Delta X(\mathbf{r})}{X(\mathbf{r})}. \tag{38}$$

Note that the left side of Eq. (38) is a function with respect to the time t', the right side is a function with respect to the position vector $\mathbf{r}$. Therefore, Eq. (38) suggests that the both sides should be equal to a constant, say, $-\beta^2$. Thus we have

$$\Gamma_{t'}(t') = -\alpha\beta^2\Gamma(t' - \tau), \quad t' > \tau, \tag{39}$$

$$\Delta X(\mathbf{r}) + \beta^2 X(\mathbf{r}) = 0. \tag{40}$$

The substitution of Eq. (37) into Eq. (35) gives

$$k\frac{\partial X(\mathbf{r})}{\partial n} + hX(\mathbf{r}) = 0, \quad \mathbf{r} \in S. \tag{41}$$

Eqs. (40) and (41) compose an eigenvalue problem. The solution gives the eigenvalues β_i $(i = 1, 2, \ldots)$ and the corresponding eigenfunctions $X_i(\mathbf{r})$ $(i = 1, 2, \ldots)$, which are called as the ith thermal natural frequency and thermal eigenmode, respectively. By using the obtained eigenfunctions, we can expand the temperature $T(\mathbf{r})$ into the following series:

$$T(\mathbf{r}, t') = \sum_{i=1}^{\infty} \Gamma^{(i)}(t')X_i(\mathbf{r}), \tag{42}$$

where $\Gamma^{(i)}$ satisfies the following ordinary differential equation:

$$\Gamma^{(i)}_{t'}(t') = -\alpha\beta_i^2\Gamma^{(i)}(t' - \tau). \tag{43}$$

The application of the initial condition (36) yields

$$\sum_{i=1}^{\infty} \Gamma^{(i)}(t')X_i(\mathbf{r}) = \phi(\mathbf{r}, t'), \quad t' \in [0, \tau]. \tag{44}$$

By the orthogonality of the eigenfunctions, we obtain

$$\Gamma^{(i)}(t') = \phi_i(t'), \quad t' \in [0, \tau], \tag{45}$$

$$\phi_i(t') = \frac{\int_R \phi(\mathbf{r}, t')X_i(\mathbf{r})\mathrm{d}R}{\int_R X_i^2(\mathbf{r})\mathrm{d}R}. \tag{46}$$

Therefore, we get the following ordinary differential equation problem:

$$\Gamma^{(i)}_{t'}(t') = -\alpha\beta_i^2\Gamma^{(i)}(t' - \tau), \tag{47}$$

$$\Gamma^{(i)}(t') = \phi_i(t'), \quad t' \in [0, \tau], \; i = 1, 2, \ldots \tag{48}$$

According to the delay ordinary differential equation theory [35], if

$$\alpha\beta_i^2\tau > 1/e, \quad e = 2.71828\ldots, \tag{49}$$

then the temperature field demonstrates the oscillation behavior. Therefore, the condition (49) can be employed to examine the occurrence of the thermal oscillation of the SPL heat conduction.

5.2. Thermal oscillation for one-dimensional SPL heat conduction

In this section, we consider the following one-dimensional heat conduction problem based on the SPL model (3):

$$\frac{1}{\alpha}\frac{\partial T(x, t')}{\partial t'} = \frac{\partial^2 T(x, t' - \tau)}{\partial x^2}, \quad 0 < x < L, \; t' > \tau, \tag{50}$$

$$T(0, t') = T(L, t') = 0, \tag{51}$$

$$T(x, t') = \sin\frac{j\pi x}{L}, \quad t' \in [0, \tau], \tag{52}$$

where L is the length of the heat conduction medium, j is a fixed positive integer. By the boundary condition (51), the temperature field can be expanded into the following series:

$$T(x, t') = \sum_{i=1}^{\infty} T_i(t')\sin\frac{i\pi x}{L}. \tag{53}$$

Substituting Eq. (53) into Eq. (50) and applying the orthogonality of the sin series $\sin\frac{i\pi x}{L}$ $(i = 1, 2, \ldots)$ give

$$\frac{\mathrm{d}T_i(t')}{\mathrm{d}t'} = -\frac{\alpha i^2\pi^2}{L^2}T_i(t' - \tau), \quad t' > \tau \; (i = 1, 2, 3, \ldots), \tag{54}$$

$$T_i(t') = 0, \; i \neq j; \quad T_j(t') = 1, \; t' \in [0, \tau]. \tag{55}$$

Obviously, $T_i(t') = 0$ $(i \neq j)$, therefore, the temperature field $T(x, t')$ can be expressed as

$$T(x, t') = T_j(t')\sin\frac{j\pi x}{L}, \tag{56}$$

while $T_j(t')$ satisfies

$$\frac{\mathrm{d}T_j(t')}{\mathrm{d}t'} = -\frac{\alpha j^2\pi^2}{L^2}T_j(t' - \tau), \quad t' > \tau, \tag{57}$$

$$T_j(t') = 1, \quad t' \in [0, \tau]. \tag{58}$$

Setting $t^* = t'/\tau$ in Eqs. (57) and (58), we have

$$\frac{\mathrm{d}T_j(t^*)}{\mathrm{d}t^*} = -\lambda T_j(t^* - 1), \quad t^* > 1, \tag{59}$$

$$T_j(t^*) = 1, \quad t^* \in [0, 1], \tag{60}$$

where $\lambda = \alpha j^2\pi^2\tau/L^2$. By utilizing the initial condition (60) and integrating Eq. (59), the values of $T_j(t^*)$ in $t^* \in (1,2]$ can be determined. Continuing this process, we obtain

$$T_j(t^*) = \sum_{i=0}^{n-1} \frac{(n-i)^i}{i!}(-1)^i\lambda^i + \sum_{j=1}^{n-2}\sum_{i=0}^{n-j-1} \frac{(n-i-j)^i}{i!}\frac{\lambda^j}{j!}(-1)^i(-1)^j\lambda^i(t^*-n)^j + \frac{\lambda^{n-1}}{(n-1)!}(-1)^{n-1}(t^*-n)^{n-1} + \frac{\lambda^n}{n!}(-1)^n(t^*-n)^n, \quad t^* \in (n, n+1], \tag{61}$$

where n is a positive integer and $t^* > 1$. By the relations $t' = t + \tau$ and $t^* = t'/\tau$, we have $t^* = t/\tau + 1$. Accordingly, the initial condition (60) may be written in the following form:

$$T_j(t) = 1, \quad t \in [-\tau, 0]. \tag{62}$$

The formula (61) enables us to determine the temperature field completely. For $\lambda = 0.2$, 0.5, 0.8 and 1.0, the dependencies of $T_j(t^*)$ on t^* are shown in Figs. 1–4. One can see that when $\lambda = 0.2$, $T_j(t^*)$ monotonously decreases with the lapse of time. For the case $\lambda = 0.5$, $T_j(t^*)$ demonstrates the thermal oscillation. This is consistent with the occurrence condition of thermal oscillation expressed in (49), that is, $\lambda > 1.0/e \approx 0.3678795$. From Figs. 3 and 4, one can see that when the λ increases further, the thermal oscillation becomes stronger.

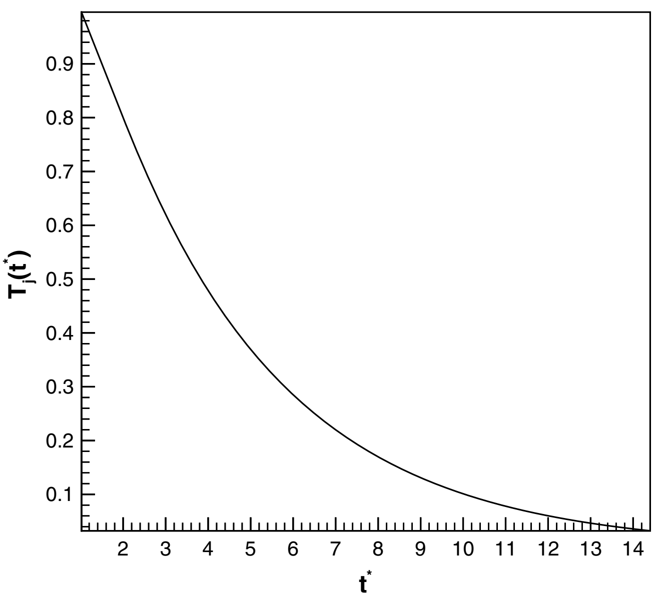

Fig. 1. $\lambda = 0.2$, the variation of $T_j(t^*)$ with respect to t^*.

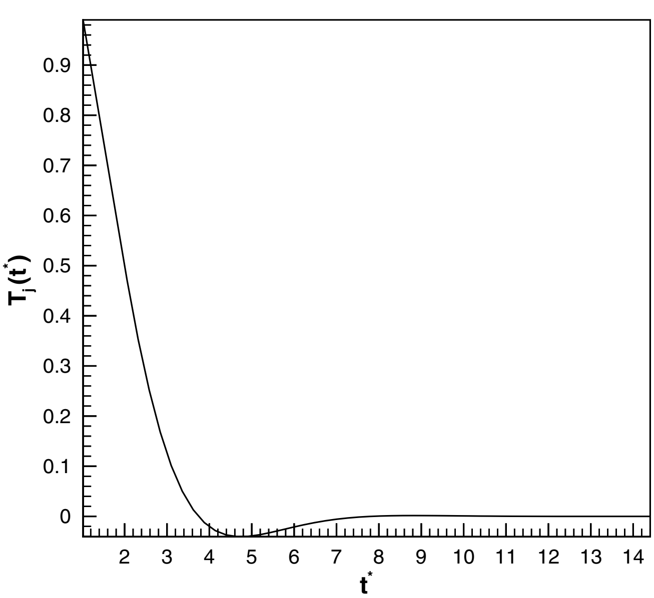

Fig. 2. $\lambda = 0.5$, the variation of $T_j(t^*)$ with respect to t^*.

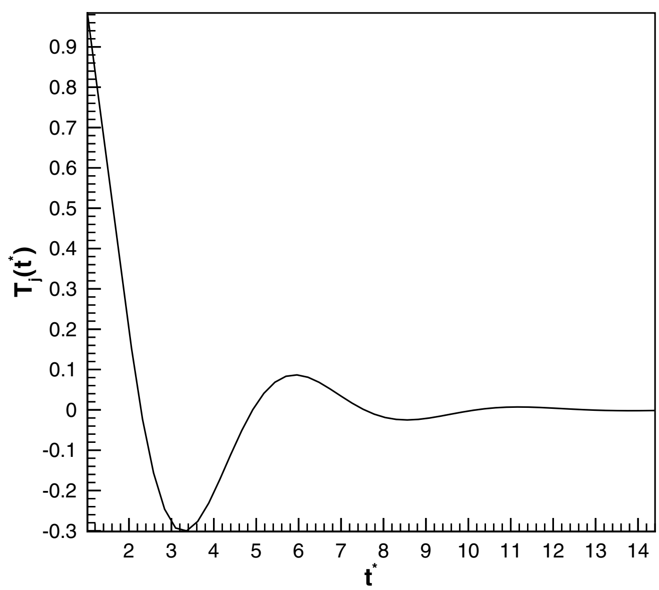

Fig. 3. $\lambda = 0.8$, the variation of $T_j(t^*)$ with respect to t^*.

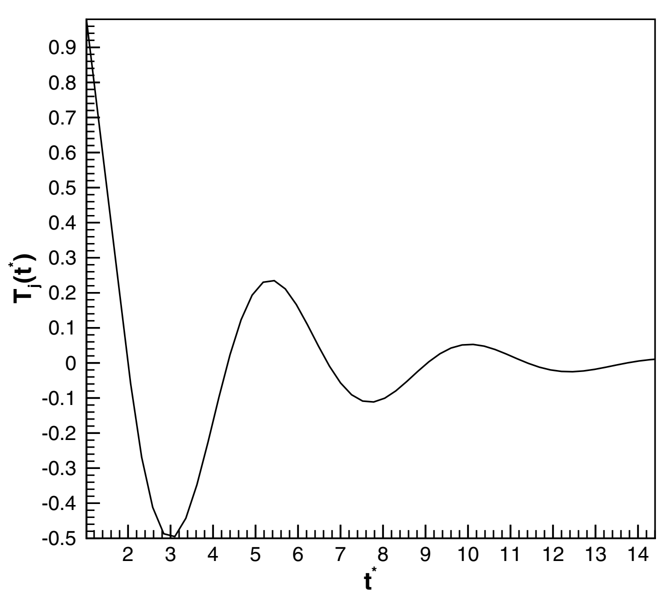

Fig. 4. $\lambda = 1.0$, the variation of $T_j(t^*)$ with respect to t^*.

6. Concluding remarks

The original SPL heat conduction model complies with the Galilean principle of relativity. It is also derivable from the Boltzmann transport equation under the assumptions that no external forces act on the heat transfer media and the system is at the quasi-equilibrium. An analytical method is developed to solve the SPL heat conduction problem. The condition for the occurrence of thermal oscillation of SPL heat conduction is established analytically. The thermal oscillation phenomenon is demonstrated by an one-dimensional SPL heat conduction problem.

It is worth noting that although the SPL heat conduction model has accounted for the temporal lagging behavior (or memory effect), it has not addressed the spatial nonlocal effect which is important in the ballistic heat transport. Recently the extended irreversible thermodynamics has been employed to establish the non-Fourier laws which are suitable for the ballistic heat transport [41–43]. The general equation for the non-equilibrium reversible–irreversible coupling and the kinetic theory were applied to the study of the ballistic–diffusive heat conduction [44]. Therefore, an important topic in the microscale heat conduction is how to improve the SPL heat conduction model and make it suitable for the ballistic heat conduction.

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